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Phase-space dynamics and quantum mechanics

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Summary. In a recent paper Deal has postulated a new dynamical equation for quantum mechanical phase-space distribution functions. We analyze the new equation and show that it may be related to the traditional standard and antistandard phase-space representations of quantum mechanics. A brief review of these and other representations is also given.

Key words: Phase-space dynamics – Correspondence rules – Quantum theory – Schrödinger equation

1. Introduction

In a recent paper on phase-space dynamics and quantum mechanics [1], Deal makes the suggestion that there is no need to invoke any of the usual postulates of quantum mechanics if one bases the description of a physical system on a phase-space distribution function, $\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t)$, with a *postulated* dynamical equation for the distribution. For an n-dimensional system with a classical Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q}, t)$ the dynamical equation for $\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t)$ is postulated to be:

$$\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t + \delta t) = h^{-n} \iint d\boldsymbol{q}_0\, d\boldsymbol{p}_0\, e^{iS/\hbar} \mathscr{D}(\boldsymbol{q}_0, \boldsymbol{p}_0, t)^* \tag{1}$$

where δt is small and:

$$S = -\Delta H\, \delta t + \Delta \boldsymbol{q} \cdot \Delta \boldsymbol{p} \tag{2}$$

with

$$\Delta H = H(\boldsymbol{p}, \boldsymbol{q}_0, t) - H(\boldsymbol{p}_0, \boldsymbol{q}, t); \qquad \Delta \boldsymbol{q} = \boldsymbol{q} - \boldsymbol{q}_0; \qquad \Delta \boldsymbol{p} = \boldsymbol{p} - \boldsymbol{p}_0. \tag{3}$$

The distribution function is assumed to be normalized:

$$\iint d\boldsymbol{q}\, d\boldsymbol{p}\, \mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) = 1 \tag{4}$$

and the average value at time t of any physically observable property $F(\boldsymbol{q}, \boldsymbol{p})$ is equal to:

$$\bar{F} = \int\int d\boldsymbol{q}\, d\boldsymbol{p}\, F(\boldsymbol{q}, \boldsymbol{p}) \mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t). \tag{5}$$

In the limit $\delta t \to 0$, Eq. (1) yields the identity:

$$\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) = h^{-n} \int\int d\boldsymbol{q}_0\, d\boldsymbol{p}_0\, e^{i\Delta \boldsymbol{q}\cdot\Delta \boldsymbol{p}/\hbar} \mathscr{D}(\boldsymbol{q}_0, \boldsymbol{p}_0, t)^* \tag{6}$$

and by taking an infinitesimal time interval $\delta t \to dt$ one gets:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) = h^{-n} \int\int d\boldsymbol{q}_0\, d\boldsymbol{p}_0\, \Delta H\, e^{i\Delta \boldsymbol{q}\cdot\Delta \boldsymbol{p}/\hbar} \mathscr{D}(\boldsymbol{q}_0, \boldsymbol{p}_0, t)^*. \tag{7}$$

Deal shows that these equations lead to (1) solutions of the form:

$$\mathscr{D}(\boldsymbol{q}, \boldsymbol{p}, t) = h^{-n/2} \psi(\boldsymbol{q}, t)^* a(\boldsymbol{p}, t)\, e^{i\boldsymbol{q}\cdot\boldsymbol{p}/\hbar} \tag{8}$$

where $\psi(\boldsymbol{q}, t)$ and $a(\boldsymbol{p}, t)$ are related as Fourier transforms, and (2) the time-dependent Schrödinger equation.

In the present paper we relate the equations suggested by Deal to the traditional phase-space formulation of quantum mechanics, in its so-called standard and antistandard versions. It is a distinguishing feature of the traditional phase-space representations that they treat states and transitions on an equal footing. Their dynamical equations will accordingly hold for both states and transitions. In accordance with this, we shall show that Deal's equations hold, not only for states, but also for certain combinations of transitions. They do not, however, hold for general transitions.

To create a background for our discussion we begin by presenting some of the main features of the traditional phase-space representations of quantum mechanics in the following section. Section 3 is devoted to a discussion of Eqs. (6) and (7) and their extensions. Finally, Sect. 4 contains our conclusions.

For simplicity of notation, we consider only one-dimensional systems throughout, the extension to several dimensions being straightforward.

2. The traditional phase-space formulation of quantum mechanics

Let $|\psi(t)\rangle$ be a normalized state vector in the Hilbert space associated with our system, and let:

$$\psi(q, t) = \langle q | \psi(t) \rangle \tag{9}$$

and

$$\phi(p, t) = \langle p | \psi(t) \rangle \tag{10}$$

be the corresponding position and momentum wave functions, in the notation of Dirac [2]. They are assumed to be normalized:

$$\int dq\, \psi(q, t)^* \psi(q, t) = \int dp\, \phi(p, t)^* \phi(p, t) = 1 \tag{11}$$

and are connected by a Fourier transformation (see, e.g. [3]):

$$\psi(q, t) = \sqrt{\frac{1}{2\pi\hbar}} \int dp\ \phi(p, t)\ e^{ipq/\hbar} \tag{12}$$

$$\phi(p, t) = \sqrt{\frac{1}{2\pi\hbar}} \int dq\ \psi(q, t)\ e^{-ipq/\hbar} \tag{13}$$

all integrations being from $-\infty$ to ∞. Then the quantity $\psi(q, t)^*\psi(q, t)$ measures the probability density in position space and $\phi(p, t)^*\phi(p, t)$ the probability density in momentum space, at time t. We may also consider the more general quantities $\psi_i(q, t)^*\psi_j(q, t)$ and $\phi_i(p, t)^*\phi_j(p, t)$ which are probability densities when i and j refer to the same state ($i = j$), and transition densities when i and j refer to different states ($i \neq j$).

Now, let $f(\theta, \tau)$ be any well behaved function for which:

$$f(0, \tau) = f(\theta, 0) = 1. \tag{14}$$

Then each such function defines a phase-space representation [4]. Thus:

$$f^s(\theta, \tau) = e^{-i\hbar\theta\tau/2} \tag{15}$$

and

$$f^a(\theta, \tau) = e^{i\hbar\theta\tau/2} \tag{16}$$

lead to the standard and antistandard representations, respectively [5], and the function:

$$f^W(\theta, \tau) = 1 \tag{17}$$

gives the Weyl–Wigner representation [6–9]. The latter is often considered to be the *canonical* phase-space representation because of its conceptually appealing properties [10–12]. The various representations are, however, equivalent and lead to the same physical predictions.

The properties of the representations characterized by Eqs. (15)–(17) are summarized in Table 1. In the following we give the general relations from which these properties may be derived.

For each representation, and for any pair of states, $|\psi_i\rangle$ and $|\psi_j\rangle$, we define the distribution function by:

$$\mathscr{D}_{ij}(q, p, t) = \frac{1}{4\pi^2} \int\int\int du\, d\theta\, d\tau\, e^{-i\theta q}\, e^{i\tau p}\, e^{i\theta u} f(\theta, \tau)\psi_i(u - \tfrac{1}{2}\hbar\tau, t)^*\psi_j(u + \tfrac{1}{2}\hbar, t). \tag{18}$$

With $f(\theta, \tau)$ as given by Eqs. (15)–(17) it reduces to the expressions $\mathscr{D}^s_{ij}$, $\mathscr{D}^a_{ij}$ and $\mathscr{D}^W_{ij}$ given in the third column of Table 1 (with the time dependence suppressed). The marginal densities are:

$$\int dp\ \mathscr{D}_{ij}(q, p, t) = \psi_i(q, t)^*\psi_j(q, t) \tag{19}$$

and

$$\int dp\ \mathscr{D}_{ij}(q, p, t) = \phi_i(p, t)^*\phi_j(p, t). \tag{20}$$

Table 1. Review of some phase-space representations

Representation	$f(\theta, \tau)$	$\mathscr{D}_{ij}(q, p)$	Operator corresponding to $q^n p^m$
Standard	$e^{-i\hbar\theta\tau/2}$	$\sqrt{\frac{1}{2\pi\hbar}}\, \psi_i(q)^* \phi_j(p)\, e^{ipq/\hbar}$	$\hat{Q}^n \hat{P}^m$
Antistandard	$e^{i\hbar\theta\tau/2}$	$\sqrt{\frac{1}{2\pi\hbar}}\, \phi_i(p)^* \psi_j(q)\, e^{-ipq/\hbar}$	$\hat{P}^m \hat{Q}^n$
Weyl–Wigner	1	$\frac{1}{2\pi}\int d\tau\, \psi_i(q - \frac{1}{2}\hbar\tau)^*$	$\frac{1}{2^n}\sum_{r=0}^{n}\binom{n}{r}\hat{Q}^r \hat{P}^m \hat{Q}^{n-r}$
		$\times\, \psi_j(q + \frac{1}{2}\hbar\tau)\, e^{-i\tau p}$	$= \frac{1}{2^m}\sum_{s=0}^{m}\binom{m}{s}\hat{P}^s \hat{Q}^n \hat{P}^{m-s}$
		$= \frac{1}{2\pi}\int d\theta\, \phi_i(p + \frac{1}{2}\hbar\theta)^*$	
		$\times\, \phi_j(p - \frac{1}{2}\hbar\theta)\, e^{i\theta q}$	

Generator of twisted product	$a(q, p)$ corresponding to operator $\hat{A}$	$a(q, p)$ corresponding to $(2\pi\hbar)^{-1}\lvert\psi_j\rangle\langle\psi_i\rvert$
$\exp\left(-i\hbar\frac{\partial}{\partial p_1}\frac{\partial}{\partial q_2}\right)$	$\int dq'\, \langle q\lvert\hat{A}\rvert q - q'\rangle\, e^{-ipq'/\hbar}$	$\mathscr{D}^a_{ij}(q, p)$
$\exp\left(i\hbar\frac{\partial}{\partial q_1}\frac{\partial}{\partial p_2}\right)$	$\int dq'\, \langle q + q'\lvert\hat{A}\rvert q\rangle\, e^{-ipq'/\hbar}$	$\mathscr{D}^s_{ij}(q, p)$
$\exp\left[\frac{i\hbar}{2}\left(\frac{\partial}{\partial q_1}\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}\frac{\partial}{\partial q_2}\right)\right]$	$\int dq'\, \langle q + \frac{1}{2}q'\lvert\hat{A}\rvert q - \frac{1}{2}q'\rangle\, e^{-ipq'/\hbar}$	$\mathscr{D}^W_{ij}(q, p)$

This is easily verified by noting that:

$$\int dy\, e^{ixy} = 2\pi\delta(x - y). \tag{21}$$

Next we construct, for each representation, a one-to-one correspondence between operators in Hilbert space and functions in phase space. Thus, if the operator $\hat{A}$ and the function $a(q, p)$ is a corresponding pair, and:

$$a(q, p) = \int\int d\theta\, d\tau\, \alpha(\theta, \tau)\, e^{i(\theta q + \tau p)} \tag{22}$$

we take $\hat{A}$ to be:

$$\begin{aligned}\hat{A} &= \int\int d\theta\, d\tau\, f(\theta, \tau)\alpha(\theta, \tau)\, e^{i(\theta\hat{Q} + \tau\hat{P})} \\ &= \frac{1}{4\pi^2}\int\int\int\int dq\, dp\, d\theta\, d\tau\, f(\theta, \tau)a(q, p)\, e^{-i(\theta q + \tau p)}\, e^{i(\theta\hat{Q} + \tau\hat{P})}\end{aligned} \tag{23}$$

where $\hat{Q}$ and $\hat{P}$ are the usual position and momentum operators satisfying the commutation relation:

$$[\hat{Q}, \hat{P}] = i\hbar. \tag{24}$$

We note that this commutation relation implies that:

$$e^{i(\theta\hat{Q}+\tau\hat{P})} = e^{i\theta\tau\hbar/2}\, e^{i\theta\hat{Q}}\, e^{i\tau\hat{P}} = e^{-i\theta\tau\hbar/2}\, e^{i\theta\hat{P}}\, e^{i\tau\hat{Q}} \tag{25}$$

as a special case of the relation:

$$e^{\hat{A}}\, e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]} \tag{26}$$

which holds when $[\hat{A}, \hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$ (see, e.g. [13]).

With the definitions given by Eqs. (18) and (23) we ensure that the following result holds in each representation:

$$\langle\psi_i(t)|\hat{A}|\psi_j(t)\rangle = \int\int dq\, dp\, a(q,p)\mathscr{D}_{ij}(q,p,t). \tag{27}$$

This is one of the central relations in the phase-space formulation of quantum mechanics. For $i = j$ it includes Eq. (5).

To proceed, let $a_1(q)$ and $a_2(p)$ be arbitrary functions of q and p respectively. It then follows, by manipulating Eq. (23), that the corresponding operators are simply $a_1(\hat{Q})$ and $a_2(\hat{P})$, for all $f(\theta, \tau)$. For more complicated functions the result depends upon the form of $f(\theta, \tau)$. As an important example we list in the fourth column of Table 1 the operator equivalent of the phase-space function $q^n p^m$, in each of the representations discussed here. Note that it is only for the Weyl–Wigner representation that the corresponding operator is Hermitian. In general, a real phase-space function $a(q,p)$ corresponds to a Hermitian operator $\hat{A}$ if and only if $f(-\theta, -\tau) = f(\theta, \tau)^*$.

The fifth column of Table 1 gives the generators of the so-called twisted product. Let, for instance, $a^s(q,p)$ and $b^s(q,p)$ be the phase-space functions corresponding to the operators $\hat{A}$ and $\hat{B}$ respectively in the standard representation. Then the phase-space function $c^s(q,p)$ corresponding to the operator $\hat{C} = \hat{A}\hat{B}$ in the same representation is:

$$c^s(q,p) = \exp\left(-i\hbar\frac{\partial}{\partial p_1}\frac{\partial}{\partial q_2}\right)a^s(q,p)b^s(q,p). \tag{28}$$

Here, the subscript 1 on a differential operator indicates that this operator acts only on the first function in the product $a^s(q,p)b^s(q,p)$. Similarly, the subscript 2 is used with operators which only act on the second function in the product.

The sixth column of the table shows how the phase-space function corresponding to an arbitrary operator $\hat{A}$ may be derived from the position-space representation $\langle q|\hat{A}|q'\rangle$ of the operator. The expressions listed are readily derived from Eq. (23) by noting that:

$$e^{i\theta\hat{Q}}|q\rangle = e^{i\theta q}|q\rangle \tag{29}$$

and

$$e^{i\tau\hat{P}}|q\rangle = |q - \tau\hbar\rangle. \tag{30}$$

The expressions of column 6 allow us to derive the phase-space function corresponding to the operator:

$$\hat{\varrho}_{ij}(t) = \frac{1}{2\pi\hbar}\,|\psi_j(t)\rangle\langle\psi_i(t)|. \tag{31}$$

The result is given in the last column of the table. It shows that this function is nothing but the distribution function $\mathscr{D}_{ij}(q, p)$ of Eq. (18), but with the important modification that the standard and antistandard distribution functions are interchanged.

We have now introduced the most fundamental relations of the phase-space formulation of quantum mechanics and can turn to a discussion of the time dependence of the distribution function $\mathscr{D}_{ij}(q, p)$.

3. The dynamical equation for the distribution function

Let $\hat{H}(\hat{Q}, \hat{P}, t)$ be the Hamiltonian for our quantum system, and let $|\psi(t)\rangle$ be any state vector. Its time dependence is given by the time-dependent Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{32}$$

To determine $\partial/\partial t\ \mathscr{D}_{ij}(q, p, t)$ we may differentiate the explicit expressions in the third column of Table 1 with respect to t, and then insert the position or momentum representation of Eq. (32). It is, however, easier and more instructive to start from the equation:

$$i\hbar \frac{\partial}{\partial t} \hat{\varrho}_{ij}(t) = \hat{H}\hat{\varrho}_{ij}(t) - \hat{\varrho}_{ij}(t)\hat{H} \tag{33}$$

obtained by applying Eq. (32) to the expression (31) for $\hat{\varrho}_{ij}(t)$, and then construct the phase-space equivalent of this equation by means of columns 5 and 7 in Table 1.

We get, for instance, by working in the Weyl–Wigner representation:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^W_{ij}(q, p, t) = 2i \sin\left[\frac{\hbar}{2}\left(\frac{\partial}{\partial q_1}\frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1}\frac{\partial}{\partial q_2}\right)\right] H^W(q, p, t)\mathscr{D}^W_{ij}(q, p, t) \tag{34}$$

where $H^W(q, p, t)$ is the Weyl–Wigner phase-space Hamiltonian. In most cases of practical interest $\hat{H}(\hat{Q}, \hat{P}, t)$ is the sum of a kinetic and potential energy term, and hence $H^W(q, p, t) = H^s(q, p, t) = H^a(q, p, t) = H(q, p, t)$, where $H(q, p, t)$ is obtained by replacing the operators $\hat{Q}$ and $\hat{P}$ in the quantum mechanical Hamiltonian by q and p. In any case the phase-space Hamiltonians can be obtained by using the information in column 4 (and perhaps column 5) of Table 1.

To obtain the dynamical equation for $\mathscr{D}^s_{ij}(q, p, t)$ we must, according to the last column of Table 1, work in the antistandard representation. We get:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^s_{ij}(q, p, t) = \left\{\exp\left(i\hbar \frac{\partial}{\partial q_1}\frac{\partial}{\partial p_2}\right) - \exp\left(i\hbar \frac{\partial}{\partial p_1}\frac{\partial}{\partial q_2}\right)\right\} H^a(q, p, t)\mathscr{D}^s_{ij}(q, p, t). \tag{35}$$

A distribution function $\mathscr{D}_{ij}(q, p, t)$ describes a state when $i = j$ and a transition when $i \neq j$. Hence, Eqs. (34) and (35) confirm the statement made in the Introduction, that the dynamical equations are the same for states and transitions. We see, in fact, that they are satisfied by any function of the type:

$$\mathscr{D}(q,p,t)=\sum_i\sum_j C_{ij}\mathscr{D}_{ij}(q,p,t) \tag{36}$$

with arbitrary coefficients.

Equations (34) and (35) may be used as they stand, but in order to make contact with Deal's equations we shall now transform Eq. (35) into an integral equation. To this end we take the expression for $\mathscr{D}^s_{ij}(q,p,t)$ from Table 1, i.e.:

$$\mathscr{D}^s_{ij}(q,p,t)=\sqrt{\frac{1}{2\pi\hbar}}\,\psi_i(q,t)^*\phi_j(p,t)\,e^{ipq/\hbar} \tag{37}$$

and replace $\psi_i(q,t)$ and $\phi_j(p,t)$ by their Fourier transforms as defined by Eqs. (11) and (12). Thus we get:

$$\mathscr{D}^s_{ij}(q,p,t)=\frac{1}{2\pi\hbar}\int\int dq_0\,dp_0\,\mathscr{D}^a_{ij}(q_0,p_0,t)\,e^{i(q-q_0)(p-p_0)/\hbar} \tag{38}$$

where

$$\mathscr{D}^a_{ij}(q,p,t)=\sqrt{\frac{1}{2\pi\hbar}}\,\phi_i(p,t)^*\psi_j(q,t)\,e^{-ipq/\hbar} \tag{39}$$

as in Table 1. In a notation similar to that of Eq. (3) we may also write:

$$\mathscr{D}^s_{ij}(q,p,t)=\frac{1}{2\pi\hbar}\int\int dq_0\,dp_0\,e^{i\Delta q\,\Delta p/\hbar}\mathscr{D}^a_{ij}(q_0,p_0,t) \tag{40}$$

and, similarly, for any function of the type (36):

$$\boxed{\mathscr{D}^s(q,p,t)=\frac{1}{2\pi\hbar}\int\int dq_0\,dp_0\,e^{i\Delta q\,\Delta p/\hbar}\mathscr{D}^a(q_0,p_0,t)}\,. \tag{41}$$

By comparing Eqs. (37) and (39) we see that:

$$\mathscr{D}^a_{ij}(q,p,t)=\mathscr{D}^s_{ji}(q,p,t)^*. \tag{42}$$

Hence, Eq. (38) may also be written:

$$\mathscr{D}^s_{ij}(q,p,t)=\frac{1}{2\pi\hbar}\int\int dq_0\,dp_0\,e^{i\Delta q\,\Delta p/\hbar}\mathscr{D}^s_{ji}(q_0,p_0,t)^*. \tag{43}$$

Because the order of the indices i and j on the right-hand side of this equation is the reverse of that on the left-hand side, a relation of the type:

$$\mathscr{D}^s(q,p,t)=\frac{1}{2\pi\hbar}\int\int dq_0\,dp_0\,e^{i\Delta q\,\Delta p/\hbar}\mathscr{D}^s(q_0,p_0,t)^* \tag{44}$$

cannot hold for arbitrary functions of the type (36). In fact, it can be true only when, in the notation of Eq. (36):

$$\left\{\sum_i\sum_j C_{ij}\mathscr{D}^s_{ij}\right\}^*=\sum_i\sum_j C_{ij}\mathscr{D}^a_{ij} \tag{45}$$

i.e.:

$$\left\{\sum_i\sum_j C_{ij}\mathscr{D}^s_{ij}\right\}^*=\sum_i\sum_j C_{ij}\mathscr{D}^{s*}_{ji}. \tag{46}$$

This relation holds only for the following functions:

$$\mathscr{D} = \begin{cases} \mathscr{D}_{ii} \\ \mathscr{D}_{ij} + \mathscr{D}_{ji} \\ i(\mathscr{D}_{ij} - \mathscr{D}_{ji}) \end{cases} \tag{47}$$

and real linear combinations of such functions.

Thus we conclude that Eq. (6), proposed by Deal, is true for any state and for some, but not all, linear combinations of transitions.

To proceed, we substitute Eq. (38) in the right-hand side of Eq. (35) and get:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^s_{ij}(q, p, t) = \frac{1}{2\pi\hbar} \int\int dq_0\, dp_0\, \mathscr{D}^a_{ij}(q_0, p_0, t) \left\{ \exp\left(i\hbar \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2}\right) - \exp\left(i\hbar \frac{\partial}{\partial p_1} \frac{\partial}{\partial q_2}\right) \right\} H^a(q, p, t)\, e^{i(q-q_0)(p-p_0)/\hbar}. \tag{48}$$

Next, we note that:

$$\begin{aligned} &\exp\left(i\hbar \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2}\right) H^a(q, p, t)\, e^{i(q-q_0)(p-p_0)/\hbar} \\ &= e^{i(p-p_0)(q-q_0)/\hbar} \exp\left\{(q_0 - q) \frac{\partial}{\partial q}\right\} H^a(q, p, t) \\ &= H^a(q_0, p, t)\, e^{i(q-q_0)(p-p_0)/\hbar} \end{aligned} \tag{49}$$

and similarly:

$$\exp\left(i\hbar \frac{\partial}{\partial p_1} \frac{\partial}{\partial q_2}\right) H^a(q, p, t)\, e^{i(q-q_0)(p-p_0)/\hbar} = H^a(q, p_0, t)\, e^{i(q-q_0)(p-p_0)/\hbar}. \tag{50}$$

Inserting the expressions (49) and (50) in Eq. (48) gives finally:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^s_{ij}(q, p, t) = \frac{1}{2\pi\hbar} \int\int dq_0\, dp_0\, \mathscr{D}^a_{ij}(q_0, p_0, t) \times \{H^a(q_0, p, t) - H^a(q, p_0, t)\}\, e^{i(q-q_0)(p-p_0)/\hbar} \tag{51}$$

or, in a notation similar to that of Eq. (3):

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^s_{ij}(q, p, t) = \frac{1}{2\pi\hbar} \int\int dq_0\, dp_0\, \Delta H^a\, e^{i\Delta q\, \Delta p/\hbar} \mathscr{D}^a_{ij}(q_0, p_0, t) \tag{52}$$

and, similarly for any function of the type of Eq. (36):

$$\boxed{i\hbar \frac{\partial}{\partial t} \mathscr{D}^s(q, p, t) = \frac{1}{2\pi\hbar} \int\int dq_0\, dp_0\, \Delta H^a\, e^{i\Delta q\, \Delta p/\hbar} \mathscr{D}^a(q_0, p_0, t)}\ . \tag{53}$$

Equation (52) may also be written:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^s_{ij}(q, p, t) = \frac{1}{2\pi\hbar} \int\int dq_0\, dp_0\, \Delta H^a\, e^{i\Delta q\, \Delta p/\hbar} \mathscr{D}^s_{ji}(q_0, p_0, t)^*. \tag{54}$$

But the relation:

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^s(q, p, t) = \frac{1}{2\pi\hbar} \iint dq_0 \, dp_0 \, \Delta H^a \, e^{i\Delta q \, \Delta p/\hbar} \mathscr{D}^s(q_0, p_0, t)^* \tag{55}$$

is only true for states and real linear combinations of functions of the type of Eq. (47). Equation (55) is seen to be identical with Eq. (7) proposed by Deal, provided that we identify his $H(q, p, t)$ with $H^a(q, p, t)$.

Thus, we have arrived at Eqs. (41) and (53) as the proper generalizations of Deal's equations. They are entirely embedded in the standard phase-space formulation of quantum mechanics.

For the sake of completeness we close this section by also listing the following equations which are similar to Eqs. (41) and (53) and may be derived along similar lines, starting from the standard representation:

$$\mathscr{D}^a(q, p, t) = \frac{1}{2\pi\hbar} \iint dq_0 \, dp_0 \, e^{-i\Delta q \, \Delta p/\hbar} \mathscr{D}^s(q_0, p_0, t) \tag{56}$$

and

$$i\hbar \frac{\partial}{\partial t} \mathscr{D}^a(q, p, t) = \frac{-1}{2\pi\hbar} \iint dq_0 \, dp_0 \, \Delta H^s \, e^{-i\Delta q \, \Delta p/\hbar} \mathscr{D}^s(q_0, p_0, t). \tag{57}$$

4. Conclusions

We have given an overview of the traditional phase-space formulation of quantum mechanics, including a table with comprehensive information concerning a few specific representations. On this background we have embedded a phase-space dynamics recently suggested by Deal in the traditional formalism.

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